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Exploring the relation between 4D and 5D BPS solutions

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ABSTRACT

Based on recent proposals linking four and five-dimensional BPS solutions, we discuss the explicit dictionary between general stationary 4D and 5D supersymmetric solutions in $N = 2$ supergravity theories with cubic prepotentials. All these solutions are completely determined in terms of the same set of harmonic functions and the same set of attractor equations. As an example, we discuss black holes and black rings in Gödel-Taub-NUT spacetime. Then we consider corrections to the 4D solutions associated with more general prepotentials and comment on analogous corrections on the 5D side.

1 Introduction and summary

Supersymmetric solutions in five dimensions come in different varieties. Among the asymptotically flat solutions, there are solutions describing black holes with or without rotation [1] – [12] as well as black rings with rotation [13] – [23]. There is also a maximally supersymmetric solution describing a Gödel universe [24]. It is, moreover, known that a rotating black hole or black ring in a Gödel spacetime also yields a supersymmetric solution [25, 26, 27]. In four dimensions, on the other hand, there are supersymmetric multi-center solutions describing a collection of extremal black holes [28] – [32].

Recently [33, 34], a very interesting relation has been established between some of these five and four-dimensional supersymmetric solutions. Based on earlier remarkable work by [24], it was shown in [33, 34] that five-dimensional black holes and black rings, when embedded in a Taub-NUT geometry, are connected to supersymmetric multi-center solutions in four dimensions. This connection is implemented by using the modulus of the Taub-NUT space to interpolate between the four and the five-dimensional description. In the vicinity of the NUT charge, the spacetime looks five-dimensional, whereas far away from the NUT the spacetime looks four-dimensional. A black hole located at the NUT will look like a five-dimensional black hole in the vicinity of the NUT, whereas it will look like a four-dimensional black hole solution far away from the NUT. Similarly, a black ring sitting at some distance from the NUT charge will, far away from the NUT, look like a four-dimensional two-center solution. One of these centers describes the location of the NUT, whereas the other center describes the position of the horizon of a four-dimensional supersymmetric black hole.

The supersymmetric solutions of minimal supergravity in five dimensions have been classified in the remarkable paper [24]. In the case when the solution possesses a timelike Killing vector, the solution is specified in terms of a hyper-Kähler four-manifold describing the spatial base geometry orthogonal to the orbits of the Killing vector field. If this base space admits a tri-holomorphic Killing vector (i.e. a Killing vector which preserves the hyper-Kähler structure), then the base space is a Gibbons-Hawking space, and the full solution is determined in terms of the Gibbons-Hawking metric and in terms of a set of harmonic functions. As described in [24], there is a dictionary which relates a subset of these five-dimensional solutions to the entire timelike class of supersymmetric solutions of four-dimensional $N = 2$ supergravity [35]. In [34], this dictionary between four and five-dimensional supersymmetric solutions has been extended to general stationary solutions of $N = 2$ supergravity theories based on cubic prepotentials.

The relation established in [33, 34] implies that any stationary five-dimensional solution of an $N = 2$ supergravity theory, when embedded in a Taub-NUT geometry, is connected to a four-dimensional stationary solution of an associated four-dimensional $N = 2$ supergravity theory. The four-dimensional solution is entirely determined in terms of a set of harmonic functions and in terms of a set of so-called stabilisation equations [28, 36, 30]. In the vicinity of a horizon these equations are also known as attractor equations, and they were first discussed in [37, 38, 39], and subsequently also in [40, 41, 42, 43]. It follows that also the five-dimensional solution is entirely determined in terms of the same set of harmonic functions and attractor equations.

In general, four-dimensional $N = 2$ supergravity theories are not simply based on cubic prepotentials, but on more general ones. This implies that four-dimensional stationary solutions will be subject to a variety of corrections, which can nevertheless be incorporated into the solution in a systematic way thanks to the attractor mechanism alluded to above. If we now assume that the connection between five-dimensional solutions in a Taub-NUT geometry and four-dimensional solutions remains valid in the presence of corrections associated with more general prepotentials, then we can determine the corrections to five-dimensional quantities such as the five-dimensional entropy. Evidence that this connection continues to hold in the presence of R^2 -corrections has recently been given in [44].

Let us explain the basic setup for connecting five to four-dimensional solutions by considering a specific example, namely supersymmetric Reissner-Nordstrom black holes in five and four dimensions. The relation between four and five-dimensional solutions can be best exhibited in a suitable coordinate system.

The line element of the five-dimensional supersymmetric Reissner-Nordstrom black hole reads

$$ds^2 = -\frac{1}{H^2}dt^2 + Hdx^m dx^m \quad , \quad A = \frac{dt}{H} \quad , \quad (1.1)$$

where H is a harmonic function given by

$$H = 1 + \left(\frac{\rho_0}{\rho}\right)^2 \quad , \quad dx^m dx^m = d\rho^2 + \frac{\rho^2}{4} \left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 \right] . \quad (1.2)$$

Here $\rho_0^2 = (16q^3 G_5^2 / \pi^2)^{1/3}$ [4], q is the electric charge and G_5 denotes Newton's constant in five dimensions. The $\sigma_i, i = 1, 2, 3$, are the left-invariant $SU(2)$ -one forms given by

$$\begin{aligned} \sigma_1 &= -\sin \psi d\theta + \cos \psi \sin \theta d\varphi \quad , \\ \sigma_2 &= \cos \psi d\theta + \sin \psi \sin \theta d\varphi \quad , \\ \sigma_3 &= d\psi + \cos \theta d\varphi \quad , \end{aligned} \quad (1.3)$$

where $\theta \in [0, \pi]$, $\varphi \in [0, 2\pi)$ and $\psi \in [0, 4\pi)$.

The entropy of this five-dimensional black hole is

$$\mathcal{S}_5 = \frac{A_5}{4G_5} = \frac{2\pi^2 \rho_0^3}{4G_5} = 2\pi\sqrt{q^3} . \quad (1.4)$$

Let us now relate this five-dimensional black hole solution to a four-dimensional black hole solution. To do so, we perform the coordinate transformation

$$\rho^2 = 4Rr \quad (1.5)$$

and obtain for (1.2)

$$H = 1 + \frac{\rho_0^2}{4Rr} , \quad dx^m dx^m = \frac{1}{N} R^2 \sigma_3^2 + N \left[dr^2 + r^2 (\sigma_1^2 + \sigma_2^2) \right] \quad (1.6)$$

with $N = R/r$. We note that since we have only done a coordinate transformation, H is still a localized harmonic function. It is now straightforward to perform a reduction over the compact coordinate $x^5 = R\psi$. In doing so, one obtains the line element for a four-dimensional supersymmetric black hole solution,

$$ds_4^2 = -e^{2U} dt^2 + e^{-2U} [dr^2 + r^2 d\Omega_2] , \quad e^{-2U} = \sqrt{NH^3} , \quad (1.7)$$

which has the finite entropy

$$\mathcal{S}_4 = \frac{A_4}{4G_4} = \frac{4\pi}{4G_4} \sqrt{R \left[\frac{\rho_0^2}{4R} \right]^3} = 2\pi\sqrt{q^3} , \quad (1.8)$$

and we have used $G_5 = 4\pi R G_4$. Thus, we obtain an exact matching of the entropies of the five and four-dimensional black hole, cf. eq. (1.4).

Next, let us consider replacing the flat four-dimensional base space in (1.6) by a Taub-NUT base, namely

$$N = \frac{R}{r} \longrightarrow N = 1 + \frac{pR}{r} , \quad (1.9)$$

where p is the NUT charge. The Taub-NUT space has a different topology than four-dimensional flat space. It has a non-trivial S_1 associated with the angle ψ , whose asymptotic radius is finite and given by R . Near $r = 0$, on the other hand, the space looks flat and four-dimensional (for $p = 1$). We note that the pole of H at $r = 0$ is a NUT fixed point of the Killing vector ∂_ψ and therefore the black hole as given in (1.6) is really localized in all spatial directions and not smeared along x^5 .

The entropy of the associated black hole, calculated in either five or four dimensions, then becomes

$$\mathcal{S}_5 = \mathcal{S}_4 = 2\pi\sqrt{pq^3} . \quad (1.10)$$

From a four-dimensional point of view, this entropy exhibits the expected quartic dependence on the charges. The dependence on p can be interpreted as a gravitational contribution to the entropy [45]. More generally, one can also add angular momentum to the five-dimensional black hole solution [2, 6], which will modify the result (1.10). Furthermore, one can also take the four-dimensional base space in (1.6) to be a multi-center Gibbons-Hawking (GH) space [46].

General $N = 2$ supergravity theories are obtained by coupling an arbitrary number of abelian vectormultiplets to $N = 2$ supergravity. Hypermultiplets can also be coupled to $N = 2$ supergravity, but since they play a spectator role in the context of stationary solutions, they will not be considered in the following. The resulting supersymmetric stationary five-dimensional solutions, when embedded in a Taub-NUT geometry, are connected to supersymmetric stationary four-dimensional solutions, as described above. The latter are determined in terms of a set of harmonic functions and in terms of so-called stabilisation equations. Therefore also a five-dimensional solution, which is connected to a four-dimensional solution, will be determined in terms of the same set of harmonic functions and stabilisation equations. The resulting dictionary will be discussed in section 2, and can be summarized as follows.

The five-dimensional $N = 2$ supergravity theory is based on the cubic prepotential function $V(Y_{5d}) = \frac{1}{6}C_{ABC}Y_{5d}^AY_{5d}^BY_{5d}^C$, where the five-dimensional scalars Y_{5d}^A are real. The associated four-dimensional $N = 2$ supergravity theory is based on the prepotential $F(Y) = -V(Y)/Y^0$, where the four-dimensional scalars $Y^I = (Y^0, Y^A)$ are complex. A four-dimensional stationary solution is characterised by real harmonic functions $(H^I ; H_I) = (N, K^A ; M, L_A)$ associated to the Y^I . The Y^I are determined in terms of these harmonic functions via the so-called stabilisation equations,

$$Y^I - \bar{Y}^I = iH^I \quad , \quad F_I(Y) - \bar{F}_I(\bar{Y}) = iH_I . \quad (1.11)$$

Solving these equations one gets

$$Y^0 = \frac{1}{2}(\phi^0 + iN) \quad , \quad Y^A = -\frac{|Y^0|}{\sqrt{N}}x^A + \frac{Y^0}{N}K^A , \quad (1.12)$$

where

$$\frac{1}{2}C_{ABC}x^Bx^C = L_A + \frac{1}{2}\frac{C_{ABC}K^BK^C}{N} \equiv \Delta_A \quad (1.13)$$

and

$$\phi^0 = e^{2U} \left(N^2 M + N L_A K^A + \frac{1}{3} C_{ABC} K^A K^B K^C \right). \quad (1.14)$$

The four-dimensional line element reads $ds^2 = -e^{2U}(dt + \vec{\omega} d\vec{x})^2 + e^{-2U} d\vec{x}^2$, with

$$\begin{aligned} e^{-4U} &= \frac{4}{9} N (x^A \Delta_A)^2 - N^{-2} \left(N^2 M + N L_A K^A + \frac{1}{3} C_{ABC} K^A K^B K^C \right)^2, \\ \nabla \times \vec{\omega} &= N \nabla M - M \nabla N + K^A \nabla L_A - L_A \nabla K^A. \end{aligned} \quad (1.15)$$

The five-dimensional solution is entirely specified in terms of these four-dimensional quantities. The five-dimensional line element reads $ds_5^2 = -f^2(dt + \omega)^2 + f^{-1}ds_{GH}^2$, where $ds_{GH}^2 = N d\vec{x}^2 + R^2 N^{-1} (d\psi + \vec{A} d\vec{x})^2$, $\nabla \times \vec{A} = R^{-1} \nabla N$ and $\omega = \omega_5 (d\psi + \vec{A} d\vec{x}) + \vec{\omega} d\vec{x}$. We have the following relations between the four and five-dimensional quantities,

$$Y_{5d}^A = 2^{1/3} x^A, \quad f^{-3/2} = \frac{2}{3} \Delta_A x^A, \quad \omega_5 = R \frac{e^{-2U} \phi^0}{N^2}. \quad (1.16)$$

This paper is organized as follows. In section 2 we derive the relation (1.16) which determines the five-dimensional stationary solution in terms of the four-dimensional stabilisation equations. In section 3 we discuss specific examples, e.g. black holes/black rings in Gödel-Taub-NUT spacetime. In section 4 we focus on R^2 -corrections to four-dimensional solutions and argue that they will affect the five-dimensional solutions via the four-dimensional stabilisation equations. We consider, in particular, the cloaking of three-charge solutions due to R^2 -interactions in four dimensions and we argue that a similar cloaking should occur for two-charge solutions in a Taub-NUT geometry in five dimensions. We also comment on the recent discussion of higher derivative corrections in four and five dimensions [47, 48].

While this work was being finalized, two papers appeared on the archive which have some overlap with ours. The paper [49] constructs supersymmetric black ring solutions in Gödel spacetime with the scalar fields valued in a symmetric space. This has some overlap with our subsection 3.2 (however, we do not restrict the scalar manifold to be a symmetric space). The paper [50] discusses the cloaking of black rings, which has some overlap with section 4.

2 Relating four and five-dimensional BPS solutions

2.1 Four-dimensional stationary BPS solutions

Four-dimensional $N = 2$ supergravity theories with n abelian vectormultiplets are based on a holomorphic prepotential function $F(Y)$ and an associated symplectic section $(Y^I, F_I(Y))$,

where $F_I = \partial F(Y)/\partial Y^I$ ($I = 0, 1, \dots, n$), for more details see [51, 52]. Stationary supersymmetric solutions in these theories are determined in terms of a set of harmonic functions (H^I, H_I) . The dependence of the scalar fields Y^I on these harmonic functions is determined via the so-called stabilisation equations [28, 30]. These were first encountered when studying the entropy of supersymmetric black hole solutions [37, 38, 39]. In the vicinity of the black hole horizon, these equations are also called attractor equations. In the following, we will briefly review the form of a stationary supersymmetric solution of these theories.

Any stationary line element can be written as

$$ds_4^2 = -e^{2U}(dt + \vec{\omega} d\vec{x})^2 + e^{-2U}d\vec{x}^2 . \quad (2.1)$$

The equations of motion for the gauge fields and the Bianchi identities are solved in terms of a symplectic vector of harmonic functions in the three coordinates \vec{x} ,

$$(H^I ; H_I) = (N, K^A ; M, L_A) \quad , \quad A = 1, \dots, n . \quad (2.2)$$

These harmonic functions are, in general, multi-center functions which are subject to a certain integrability condition (c.f. eq. (2.7)). Additional constraints may result by demanding absence of closed timelike curves (CTCs), c.f. eqs. (2.46), (2.47). The harmonic functions can, however, also be taken to be linear, in which case they are not related to sources. We will discuss various choices of harmonic functions in the next section.

For the above stationary solution, the symplectic section (Y^I, F_I) is determined in terms of the harmonic functions via the so-called stabilisation equations,

$$Y^0 - \bar{Y}^0 = iN \quad , \quad Y^A - \bar{Y}^A = iK^A , \quad (2.3)$$

$$F_0 - \bar{F}_0 = iM \quad , \quad F_A - \bar{F}_A = iL_A . \quad (2.4)$$

The line element is determined by

$$e^{-2U} = i(\bar{Y}^I F_I - \bar{F}_I Y^I) , \quad (2.5)$$

$$\nabla \times \vec{\omega} = N\nabla M - M\nabla N + K^A \nabla L_A - L_A \nabla K^A . \quad (2.6)$$

Sources for the harmonic functions yield an integrability constraint for equation (2.6). By contracting it with another derivative one finds [29],

$$0 = N\Delta M - M\Delta N + K^A \Delta L_A - L_A \Delta K^A , \quad (2.7)$$

where Δ is the three-dimensional Laplacian. The gauge field one-forms are given by [30]

$$A^I = e^{2U}(Y^I + \bar{Y}^I)(dt + \vec{\omega} d\vec{x}) - \vec{\alpha}^I d\vec{x} \quad , \quad \nabla \times \vec{\alpha}^I = \nabla H^I . \quad (2.8)$$

In the following, we will consider stationary solutions based on the cubic prepotential

$$F(Y) = \frac{D_{ABC} Y^A Y^B Y^C}{Y^0} . \quad (2.9)$$

Inserting this into (2.5) yields

$$e^{-2U} = |Y^0|^2 \mathcal{V}^3 , \quad (2.10)$$

where

$$\mathcal{V}^3 = -i D_{ABC} (z - \bar{z})^A (z - \bar{z})^B (z - \bar{z})^C \quad (2.11)$$

and $z^A = Y^A / Y^0$.

Next, we solve the stabilisation equations (2.3) and (2.4) for the cubic prepotential (2.9). In doing so we will closely follow [53]. The equations (2.3) are solved by

$$Y^0 = \frac{1}{2}(\phi^0 + i N) \quad , \quad Y^A = -\frac{|Y^0|}{\sqrt{N}} x^A + \frac{Y^0}{N} K^A . \quad (2.12)$$

The real quantities x^A are determined via the stabilisation equations $F_A - \bar{F}_A = i L_A$, which read

$$-3 D_{ABC} x^B x^C = L_A - 3 \frac{D_A}{N} \equiv \Delta_A . \quad (2.13)$$

Here we introduced

$$D \equiv D_{ABC} K^A K^B K^C , \quad D_A \equiv D_{ABC} K^B K^C , \quad D_{AB} \equiv D_{ABC} K^C . \quad (2.14)$$

It follows that

$$z^A - \bar{z}^A = i \frac{\sqrt{N}}{|Y^0|} x^A \quad , \quad \mathcal{V}^3 = -\frac{N^{3/2}}{|Y^0|^3} D_{ABC} x^A x^B x^C , \quad (2.15)$$

with x^A completely determined in terms of the harmonic functions (2.2). The x^A are taken to be positive so that $T^A + \bar{T}^A = \sqrt{N} |Y^0|^{-1} x^A > 0$, where $T^A = -i z^A$.

The remaining stabilisation equation $F_0 - \bar{F}_0 = i M$ can then be written as

$$NM = -\sqrt{N} D_{ABC} x^A x^B x^C \frac{\phi^0}{|Y^0|} + 3 D_{AB} x^A x^B - \frac{D}{N} , \quad (2.16)$$

which yields

$$|Y^0|^2 = \frac{N^3 (x^A \Delta_A)^2}{4N (x^A \Delta_A)^2 - 9 N^{-2} (N^2 M + N L_A K^A - 2 D)^2} \quad (2.17)$$

and

$$e^{-2U} \phi^0 = N^2 M + N H_A K^A - 2D . \quad (2.18)$$

For the metric function (2.10) one therefore obtains

$$e^{-4U} = \frac{4}{9} N (x^A \Delta_A)^2 - N^{-2} \left(N^2 M + N L_A K^A - 2D \right)^2 . \quad (2.19)$$

This completely determines the metric function e^{-2U} and the scalar fields z^A and Y^0 in terms of the set of harmonic functions (2.2). The metric function e^{-2U} and the scalar fields appear to be ill-defined when $N \rightarrow 0$, but this is only an artifact of the parameterization in terms of the quantities x^A . This can be verified by performing a power series expansion in N . This can also be directly checked for specific examples. For instance, consider the simple example of the so-called STU-model, which is determined by the prepotential $F(Y) = -Y^1 Y^2 Y^3 / Y^0$. For this model, the metric function is obtained as [54, 55]

$$\begin{aligned} e^{-4U} = & 4N L_1 L_2 L_3 - 4M K^1 K^2 K^3 - (MN + L_1 K^1 + L_2 K^2 + L_3 K^3)^2 \\ & + 4(L_1 K^1 L_2 K^2 + L_1 K^1 L_3 K^3 + L_2 K^2 L_3 K^3) . \end{aligned} \quad (2.20)$$

The case discussed in the introduction (c.f. eq. (1.7)) is obtained by equalizing the harmonic functions L_A and setting $K^A = M = 0$.

2.2 Five-dimensional stationary BPS solutions

Five-dimensional $N = 2$ supergravity theories with n abelian vectormultiplets are based on a real cubic prepotential function $V(X_{5d})$, where the X_{5d}^A denote real scalar fields ($A = 1, \dots, n$), for more details see [56]. These scalar fields are constrained via

$$V(X_{5d}) = -D_{ABC} X_{5d}^A X_{5d}^B X_{5d}^C = 1 \quad , \quad D_{ABC} = -\frac{1}{6} C_{ABC} . \quad (2.21)$$

Stationary supersymmetric solutions in these theories have been constructed in the nice paper [19]. The five-dimensional line element is given by

$$ds_5^2 = -f^2(dt + \omega)^2 + f^{-1} ds_{HK}^2 , \quad (2.22)$$

where ds_{HK}^2 is the line element for any hyper-Kähler space. In order to be able to connect five-dimensional solutions to the four-dimensional solutions discussed in the previous subsection, we take the hyper-Kähler space to admit a tri-holomorphic isometry, in which case its line element can be written in terms of a Gibbons-Hawking metric [46],

$$ds_{HK}^2 = N d\vec{x}^2 + R^2 N^{-1} (d\psi + \vec{A} d\vec{x})^2 , \quad (2.23)$$

$$\nabla \times \vec{A} = R^{-1} \nabla N , \quad (2.24)$$

where the modulus R determines the radius of the S^1 circle parameterized by the coordinate ψ . The function N can be any harmonic function of \vec{x} . For the Taub-NUT space, it is given by

$$N = 1 + \frac{p^0 R}{r} \quad , \quad \psi \sim \psi + 4\pi p^0 . \quad (2.25)$$

The periodicity in ψ ensures the absence of conical singularities (Dirac-Misner strings).

For a Gibbons-Hawking base space, the one-form ω in (2.22) is given by [19]

$$\omega = \omega_5 (d\psi + \vec{A}d\vec{x}) + \vec{\omega}d\vec{x} , \quad (2.26)$$

where

$$\nabla \times \vec{\omega} = N \nabla M - M \nabla N + K^A \nabla L_A - L_A \nabla K^A , \quad (2.27)$$

$$\omega_5 = R N^{-2} (N^2 M + N L_A K^A - 2 D) . \quad (2.28)$$

The metric function f , on the other hand, is determined as follows [11, 19]. One introduces the rescaled variables

$$Y_{5d}^A = f^{-1/2} X_{5d}^A . \quad (2.29)$$

Then

$$V(Y_{5d}) = \frac{1}{6} C_{ABC} Y_{5d}^A Y_{5d}^B Y_{5d}^C = f^{-3/2} . \quad (2.30)$$

The Y_{5d}^A are subject to the five-dimensional stabilisation equations, as obtained in^a [11, 19]

$$\frac{1}{2} C_{ABC} Y_{5d}^B Y_{5d}^C = 2^{2/3} \left(L_A + \frac{1}{2} \frac{C_{ABC} K^B K^C}{N} \right) . \quad (2.31)$$

It follows that

$$f^{-3/2} = \frac{2^{2/3}}{3} \left(L_A + \frac{1}{2} \frac{C_{ABC} K^B K^C}{N} \right) Y_{5d}^A . \quad (2.32)$$

Observe that, for the simple case given in (2.20), the function f factorizes yielding

$$f^{-3} = 4 \left(L_1 + \frac{K^2 K^3}{N} \right) \left(L_2 + \frac{K^3 K^1}{N} \right) \left(L_3 + \frac{K^1 K^2}{N} \right) . \quad (2.33)$$

^aThe conventions used in [19] differ from ours in the following way: $L \rightarrow \frac{2^{2/3}}{3} L$, $K \rightarrow -2^{4/3} K$.

2.3 Dictionary

We are now in a position to discuss the dictionary between five and four-dimensional stationary solutions. This dictionary has also been given in [34] using different conventions.

Comparing the five-dimensional stabilisation equations (2.31) with the four-dimensional counterpart (2.13), we see that they are identical and therefore

$$Y_{5d}^A = 2^{1/3} x^A \quad , \quad f^{-3/2} = \frac{2}{3} \Delta_A x^A . \quad (2.34)$$

The equations for $\vec{\omega}$, (2.27) and (2.6), are also identical. Comparing (2.28) with (2.18) yields the relation

$$\omega_5 = R N^{-2} e^{-2U} \phi^0 . \quad (2.35)$$

The $U(1)$ isometry of the Gibbons-Hawking metric (2.23) is generated by ∂_ψ and hence we can perform the Kaluza-Klein reduction over ψ to four dimensions. Following standard formulae, we write the five-dimensional line element (2.22), the five-dimensional gauge field one-forms and the five-dimensional scalar fields as

$$ds_5^2 = e^{2\phi} ds_4^2 + e^{-4\phi} (R d\psi - A_4^0)^2 , \quad (2.36)$$

$$A_{5d}^A = A_4^A + \text{Re} z^A (R d\psi - A_4^0) , \quad (2.37)$$

$$X_{5d}^A = -i e^{2\phi} (z^A - \bar{z}^A) , \quad (2.38)$$

where ds_4^2 denotes the four-dimensional line element given in (2.1), (A_4^0, A_4^A) are the four-dimensional gauge field one-forms and the z^A denote the four-dimensional scalar fields. Using (2.12), the real part of z^A is computed to give

$$z^A + \bar{z}^A = i \frac{\phi^0}{N} (z^A - \bar{z}^A) + 2 \frac{K^A}{N} = -\frac{\phi^0}{|Y^0|} \frac{x^A}{\sqrt{N}} + 2 \frac{K^A}{N} . \quad (2.39)$$

Comparing (2.36) with (2.22) we obtain

$$A_4^0 = \frac{\omega_5}{R} N^2 e^{4U} (dt + \vec{\omega} d\vec{x}) - R \vec{A} d\vec{x} . \quad (2.40)$$

Using (2.35) as well as (2.24) we see that (2.40) is in full agreement with (2.8). In addition, by comparing (2.36) with (2.22) we also obtain

$$e^{-4\phi} = N^{-1} f^{-1} - \left(\frac{f \omega_5}{R} \right)^2 = \frac{f^2}{N^2} e^{-4U} . \quad (2.41)$$

Using (2.17) and (2.34) we find

$$|Y^0|^2 = e^{4U} N^3 f^{-3} , \quad (2.42)$$

and using (2.10) we establish

$$e^{-4\phi} = \mathcal{V}^2 . \quad (2.43)$$

Using (2.34), (2.35) and (2.18) it can be checked that the expression for e^{-4U} , as computed from (2.41), fully agrees with the expression (2.19). And finally, using (2.17), (2.19), (2.10) and (2.34) we obtain

$$\frac{|Y^0|}{\sqrt{N}} = 2^{-1/3} f^{-1/2} \mathcal{V}^{-1} . \quad (2.44)$$

Together with (2.15) and (2.34) we establish

$$Y_{5d}^A = -i f^{-1/2} \mathcal{V}^{-1} (z^A - \bar{z}^A) , \quad (2.45)$$

which is in precise agreement with (2.38).

Consistency of the solution requires the following positivity constraints to be satisfied, namely

$$e^{-4\phi} = (Nf)^{-1} - \left(\frac{f\omega_5}{R} \right)^2 = \frac{f^2}{N^2} e^{-4U} > 0 , \quad (2.46)$$

$$\det(-e^{2U} \omega_m \omega_n + e^{-2U} \delta_{mn}) = e^{-2U} (-|\omega|^2 + e^{-4U}) > 0 , \quad (2.47)$$

where $|\omega|^2 = \delta^{mn} \omega_m \omega_n$. In addition, for a given supergravity solution (for instance black holes), one has to investigate whether Dirac-Misner strings are present. Demanding their absence may enforce the additional constraint $\vec{\omega} = 0$, at least near the centers of the solution and also asymptotically.

To summarise, we see that the five-dimensional solution, which is expressed in terms of $f, \omega_5, \vec{\omega}, Y_{5d}^A$ and A_{5d}^A , is entirely expressed in terms of the harmonic functions (2.2) and in terms of the four-dimensional variables x^A and Y^0 . The latter are determined by solving the four-dimensional stabilisation equations (2.3) and (2.4). A related discussion on attractors and five-dimensional solutions has appeared in [57, 58]. Observe that, even though there are n five-dimensional real scalar fields Y_{5d}^A , the solution is expressed in terms of $2(n+1)$ harmonic functions [19].

The four-dimensional stationary solutions are subject to a variety of corrections associated to additional non-cubic terms in the prepotential function (2.9). These corrections can be computed in a systematic way thanks to the stabilisation equations, which continue to hold [30]. If, in the presence of these corrections, the connection between four and five-dimensional solutions in Taub-NUT geometries continues to hold, then the four-dimensional stabilisation equations provide a powerful tool for computing corrections to five-dimensional quantities, such as the entropy of a five-dimensional black hole in a Taub-NUT geometry.

3 Examples

Let us now discuss various examples in detail. Each will correspond to a specific choice of the harmonic functions introduced in (2.2).

3.1 Black holes and black rings and their entropy

The simplest examples are provided by single-center black holes, which are described by the harmonic functions

$$N = n + p^0 \frac{R}{r} \quad , \quad K^A = h^A + p^A \frac{(RG_4)^{1/3}}{r} \quad , \quad (3.1)$$

$$M = m + q_0 \frac{G_4}{Rr} \quad , \quad L_A = h_A + q_A \frac{(RG_4)^{2/3}}{Rr} \quad . \quad (3.2)$$

The integrability constraint (2.7) becomes (here we set $G_4 = R^2$)

$$mp^0 - nq_0 + h_A p^A - h^A q_A = 0 \quad . \quad (3.3)$$

The symplectic vector $(n, h^A; m, h_A)$ comprising the constant parameters and the symplectic charge vector $(p^0, p^A; q_0, q_A)$ are therefore mutually local. In addition, one also has the constraint $e^{-2U} \rightarrow 1$ as $r \rightarrow \infty$. Thus, there are two conditions on the constant parameters. The number of free parameters is therefore given by twice the number n of abelian vector multiplets. In our conventions, the charges $(p^0, p^A; q_0, q_A)$ are integer valued and the dimensions are absorbed into the factors of G_4 and R . That the charges q_A and p^0 are quantized in units of $(1/R)^{1/3}$ and R , respectively, is already manifest in the example discussed in the introduction (c.f. (1.8)). This fits with the general expectation that electric/magnetic charges are associated with momentum/winding modes along the circle in the ψ direction. The correct powers of G_4 and R in M and K^A are then deduced from consistency.

From the four-dimensional point of view, all charges are on equal footing and defined as asymptotic surface integrals, as usual. In five dimensions, on the other hand, the q_A are the usual electric charges of the black hole, whereas the p^A appear as dipole charges. The charges q_0 and p^0 are on a different footing, namely p^0 is the NUT charge, whereas q_0 is related to the angular momentum of the black hole. The latter gets corrected by the other charges that enter in ω_5 .

It is well known that there are CTCs hidden behind the five-dimensional black hole horizon and that this solution becomes pathological in the over-rotating case [12, 59, 60, 61, 62, 63, 64]. In the example given in (2.20), the latter is manifest and occurs when the function

e^{-4U} becomes vanishing at the horizon. This happens when q_0 becomes large enough. In four dimensions this corresponds to a curvature singularity. In five dimensions, on the other hand, the function f remains finite (since x^A and Δ_A are independent of M) (c.f. (2.33)), but ω_5 becomes large and renders the ∂_ψ -circle timelike (c.f. (2.41), (2.36)). At the point where the radius of the circle vanishes, the four-dimensional solution is singular. A consequence of the vanishing of e^{-2U} is that \mathcal{V} also vanishes (c.f. 2.43) and hence, the scalar fields $z^A - \bar{z}^A$ also go to zero. This implies that the scalar fields are deep in the interior of the Kähler cone. In this regime instanton corrections to the prepotential become relevant and they have the property of regularising the solution and rendering the entropy finite. This has been discussed in [65].

For a generic choice of charges, the black hole has a regular horizon and the entropy, calculated in the four-dimensional setting as well as in the five-dimensional approach match exactly. If we denote the two-dimensional horizon area in four dimensions by A_4 , the entropy, given by the Bekenstein-Hawking formula reads

$$\mathcal{S}_4 = \frac{A_4}{4G_4} = \frac{4\pi}{4G_4} (e^{-2U} r^2)|_{r=0} = \pi e^{-2U_0} , \quad (3.4)$$

where e^{-2U_0} is given by (2.19), but with all harmonic functions replaced by their quantized charges, i.e. $(N, K^A; M, L_A) \rightarrow (p^0, p^A; q_0, q_A)$. In five dimensions the entropy is related to the three-dimensional area A_5 of the horizon parameterized by the three coordinates (ψ, θ, φ) . Inspection of (2.36), (2.22) and (2.25) shows that A_5 is given by

$$A_5 = 16\pi^2 R p^0 e^{-2\phi} f^{-1} N r^2|_{r=0} = 16\pi^2 R p^0 e^{-2U_0} , \quad (3.5)$$

where we have used (2.41). The associated entropy is then given by

$$\mathcal{S}_5 = \frac{A_5}{4G_5} = \pi e^{-2U_0} , \quad (3.6)$$

where we used $G_5 = (4\pi R p^0) G_4$. Hence [34]

$$\mathcal{S}_5 = \mathcal{S}_4 = 2\pi \sqrt{p^0 \left(\frac{\tilde{x}^A \tilde{\Delta}_A}{3} \right)^2 - (p^0)^2 J^2} , \quad (3.7)$$

where $\tilde{x}^A \tilde{\Delta}_A$ equals $x^A \Delta_A$ with the harmonic functions replaced by the charges, and

$$2J = q_0 + \frac{p^A q_A}{p^0} - 2 \frac{D_{ABC} p^A p^B p^C}{(p^0)^2} = \phi^0 e^{-2U} N^{-2} R G_4^{-1} r|_{r=0} . \quad (3.8)$$

Observe that the pole in p^0 is only an artifact of the parameterization in terms of the x^A .

The single-center solution can be generalized to a multi-center one by considering more general harmonic functions,

$$N = n + \sum_i p_i^0 \frac{R}{r_i} \quad , \quad K^A = h^A + \sum_i p_i^A \frac{(RG_4)^{1/3}}{r_i} \quad , \quad (3.9)$$

$$M = m + \sum_i q_0^i \frac{G_4}{R r_i} \quad , \quad L_A = h_A + \sum_i q_A^i \frac{(RG_4)^{2/3}}{R r_i} \quad , \quad (3.10)$$

where $r_i = |\vec{x} - \vec{x}_i|$. Inserting these functions into the integrability constraint (2.7) gives

$$\sum_i (N q_0^i - M p_i^0 + K^A q_A^i - L_A p_i^A) \delta^{(3)}(\vec{x} - \vec{x}_i) = 0 \quad , \quad (3.11)$$

here we have set $G_4 = R^2$ for simplicity. By integrating these equations without putting any constraints on the positions \vec{x}_i of the centers, we obtain the following conditions

$$n q_0^j - m p_j^0 + h^A q_A^j - h_A p_j^A = 0 \quad \forall \quad j \quad , \quad (3.12)$$

$$p_i^0 q_0^j - q_0^i p_j^0 + p_i^A q_A^j - q_A^i p_j^A = 0 \quad \forall \quad i \neq j \quad . \quad (3.13)$$

These conditions imply that the symplectic charge vectors $(p_i^0, p_i^A; q_0^i, q_A^i)$ and the symplectic vector $(n, h^A; m, h_A)$ are all mutually local. This severely constrains the parameters and the charges. On the other hand, eq. (3.11) can also be seen as a constraint on the positions \vec{x}_i [29, 34]. This gives the relation

$$N_i q_0^i - M^i p_i^0 + K_i^A q_A^i - L_A^i p_i^A = 0 \quad , \quad (3.14)$$

where $N_i \equiv N|_{\vec{x}=\vec{x}_i}$, $M^i \equiv M|_{\vec{x}=\vec{x}_i}$, etc. By varying \vec{x}_i one continuously changes the values of N_i , M^i , etc, and hence these equations can always be solved.

For a generic choice of charges, each center describes a black hole, from a four dimensional point of view. From a five-dimensional point of view, these centers may either correspond to black holes or to black rings [34].

A particular four-dimensional two-center solution is connected to the five-dimensional BPS black ring solution, which has attracted much attention recently [13, 14, 15, 16, 18, 19, 66, 67, 20, 21, 47, 68]. This solution corresponds to the following choice of harmonic function^b

$$\begin{aligned} K^A &= \frac{p^A}{\Sigma} \quad , \quad L_A = h_A + \frac{q_A}{\Sigma} \quad , \\ M &= -h_A p^A \left(1 - \frac{a}{\Sigma}\right) \quad , \quad N = n + \frac{1}{r} \quad , \\ \Sigma &= |\vec{x} - \vec{x}_0| = \sqrt{r^2 + a^2 + 2ra \cos \theta} \quad , \end{aligned} \quad (3.15)$$

^bIn order to simplify the notation we set $G_4 = R = 1$.

where $\vec{x}_0 = (0, 0, -a)$. Therefore, the harmonic function N is sourced at the center $r = 0$, whereas the other harmonic functions are sourced at the location of the black ring \vec{x}_0 .

This choice of harmonic functions describes a black ring located at $\vec{x} = \vec{x}_0$ with a horizon geometry $S^1 \times S^2$. This geometry is not (a deformed) S^3 , because the Gibbons-Hawking fibre is trivial at $\vec{x} = \vec{x}_0$. Hence one can always find a coordinate system so that $d\psi + \vec{A}d\vec{x} = d\psi$ at the position of the ring, which results in a factorized horizon geometry (note that there is coordinate singularity at the horizon [15, 19]).

In order to calculate $\vec{\omega} d\vec{x}$ we can use the expressions derived in [21], which in our notation become

$$\nabla \times \vec{\omega}^{(1)} = \nabla \frac{1}{r} \quad \text{with} \quad \vec{\omega}^{(1)} d\vec{x} = \cos \theta d\varphi, \quad (3.16)$$

$$\nabla \times \vec{\omega}^{(2)} = \nabla \frac{1}{\Sigma} \quad \text{with} \quad \vec{\omega}^{(2)} d\vec{x} = \frac{r \cos \theta + a}{\Sigma} d\varphi, \quad (3.17)$$

$$\nabla \times \vec{\omega}^{(3)} = \frac{1}{\Sigma} \nabla \frac{1}{r} - \frac{1}{r} \nabla \frac{1}{\Sigma} \quad \text{with} \quad \vec{\omega}^{(3)} d\vec{x} = \left(\frac{r/a + \cos \theta}{\Sigma} - \frac{1}{a} \right) d\varphi, \quad (3.18)$$

and hence, for the harmonic functions in (3.15), $\vec{\omega} d\vec{x}$ is given by

$$\vec{\omega} d\vec{x} = h_{AP}^A \left([\cos \theta + 1] \left[1 - \frac{a+r}{\Sigma} \right] + na \left[\frac{r \cos \theta + a}{\Sigma} - 1 \right] \right) d\varphi. \quad (3.19)$$

At $r = 0$, the quantities $M, \vec{\omega}$ and ω_5 vanish. The behavior at $r \rightarrow \infty$ depends crucially on the Taub-NUT parameter n . If the constant part is not present, as for the original black ring solution, $\vec{\omega}$ and ω_5 vanish asymptotically, but for $n \neq 0$, both quantities remain finite. This raises the issue of the appearance of Dirac-Misner strings, which can however be avoided if we choose the parameter n in such a way that $\omega = \omega_5(d\psi + \cos \theta d\varphi) + \vec{\omega} d\vec{x}$ becomes trivial at infinity. On the other hand the corresponding four-dimensional solution is still pathological because $\vec{\omega} d\vec{x} \simeq \cos \theta d\varphi$ for $r \rightarrow \infty$, and hence will have Dirac-Misner strings. This behavior may perhaps be avoided if one adds further appropriate constant parts to the harmonic functions, for example to M .

The black ring solution corresponds to a two-center solution in four dimensions, with the center at \vec{x}_0 describing a four-dimensional black hole [34]. We can compute its entropy by replacing the harmonic functions $(N, H^A; M, H_A)$ in (2.19) by the charges $(p^0, p^A, q_0, q_A) = (1, p^A, ah_{AP}^A, q_A)$. For the example given in (2.20), we find

$$\begin{aligned} \mathcal{S}_4 &= \pi (e^{-2U} |\vec{x} - \vec{x}_0|^2) |_{\vec{x}=\vec{x}_0} \\ &= 2\pi \sqrt{(q_1 p^1 q_2 p^2 + q_1 p^1 q_3 p^3 + q_2 p^2 q_3 p^3) - \frac{(q_{AP}^A)^2}{4} - a(h_{AP}^A) p^1 p^2 p^3}, \end{aligned} \quad (3.20)$$

which is in agreement with the expression for the black ring entropy given in [19].

The horizon of the black ring solution has geometry $S^1 \times S^2$ with an associated area of $2\pi l$ and $\pi\nu^2$, respectively [19]. This horizon geometry is the same as the one of an extremal BTZ black hole times a two-sphere [15]. The BTZ black hole has entropy $\mathcal{S}_3 = \pi l/(2G_3)$. Using $G_3^{-1} = \pi\nu^2 G_5^{-1}$ gives $\mathcal{S}_3 = \pi^2 l\nu^2/(2G_5)$, which is the entropy of the black ring [19]. On the other hand, the five-dimensional black ring in a Taub-NUT geometry is connected to a four-dimensional black hole, as discussed above. We therefore have the equality

$$\mathcal{S}_5 = \mathcal{S}_4 = \mathcal{S}_3 \quad (3.21)$$

for the entropies.

One can, of course, also construct general multi-center solutions in five dimensions [19]. A necessary condition for obtaining a black ring instead of a black hole at a given center is the absence of a source for N at that point [34]. Upon reduction to four dimensions all these solutions become multi-center black holes – a black ring can never become a single-center black hole. From the four-dimensional perspective, one can generate a five-dimensional black ring by moving the entire NUT charge p^0 of a black hole to a different position. The original black hole is generically still regular, but there is a naked singularity at the position of the NUT charge. In five dimensions this is a coordinate singularity, and this process describes the topology change from $S^3 \rightarrow S^1 \times S^2$.

3.2 Black holes and black rings in Gödel-Taub-NUT spacetime

A maximally supersymmetric Gödel solution in five dimensions has been obtained in [24]. Its metric function f is constant and the two-form $d\omega$ is anti-self-dual. This solution has CTCs at every point in spacetime, similar to what happens for the four-dimensional rotating Gödel universe. CTCs at every point in spacetime occur when a region of spacetime, where CTCs exist, is not separated from the rest of spacetime by a black hole or a cosmological horizon [69]. Various aspects of this supersymmetric solution have been discussed in the literature [70, 71, 72, 73, 74].

Supersymmetric solutions describing either a black hole or a black ring in a Gödel universe were constructed in [25, 26, 27] in minimal five-dimensional supergravity. Here we will construct black hole/black ring solutions in Gödel-Taub-NUT spacetime arising in five-dimensional supergravity theories with abelian vectormultiplets.

The Gödel solution of [24] corresponds to the following choice for the harmonic functions (2.2),

$$M = \mathcal{G} z = \mathcal{G} r \cos \theta , \quad N = \frac{R}{r} , \quad L_A = h_A = \text{const} , \quad K^A = 0 , \quad (3.22)$$

with $\mathcal{G} = \text{const.}$ This ensures that the Y_{5d}^A and f are constant (c.f. (2.31), (2.30)). The above choice of N describes a flat four-dimensional base space. The associated five-dimensional line element reads

$$ds_5^2 = -\left(dt + \mathcal{G} R r [d\varphi + \cos \theta d\psi]\right)^2 + \frac{R^2}{N} (d\psi + \cos \theta d\varphi)^2 + N (dr^2 + r^2 d\Omega_2) . \quad (3.23)$$

For large values of r the timelike $U(1)$ fibration becomes dominant, resulting in the appearance of CTCs, ie. ∂_φ as well as ∂_ψ are then inside the future directed lightcone.

The set of harmonic functions in (3.22) describing the Gödel deformation \mathcal{G} can be superimposed with the set of harmonic functions in (3.1), (3.2) and (3.15). The resulting solutions then describe either a black hole or a black ring in a Gödel-Taub-NUT spacetime.

Let us first construct a black hole solution in a Gödel-Taub-NUT spacetime. Setting $G_4 = R^2$ for convenience, we consider the following set of harmonic functions,

$$\begin{aligned} N &= \mathcal{G}_2 r \cos \theta + n + p^0 \frac{R}{r} , & K^A &= h^A + p^A \frac{R}{r} , \\ M &= \mathcal{G}_1 r \cos \theta + m + q_0 \frac{R}{r} , & L_A &= h_A + q_A \frac{R}{r} , \end{aligned} \quad (3.24)$$

where, for later convenience, we also allow for a Gödel deformation of N parameterized by \mathcal{G}_2 . As in (3.16) – (3.18) we will first give the different contributions to $\vec{\omega}$ that involve a Gödel deformation,

$$\nabla \times \vec{\omega}^{(4)} = \nabla (r \cos \theta) \quad \text{with} \quad \vec{\omega}^{(4)} d\vec{x} = \frac{1}{2} r^2 \sin^2 \theta d\varphi , \quad (3.25)$$

$$\nabla \times \vec{\omega}^{(5)} = \frac{1}{r} \nabla (r \cos \theta) - r \cos \theta \nabla \frac{1}{r} \quad \text{with} \quad \vec{\omega}^{(5)} d\vec{x} = r \sin^2 \theta d\varphi , \quad (3.26)$$

$$\nabla \times \vec{\omega}^{(6)} = \frac{1}{\Sigma} \nabla (r \cos \theta) - (a + r \cos \theta) \nabla \frac{1}{\Sigma} \quad \text{with} \quad \vec{\omega}^{(6)} d\vec{x} = \frac{r^2}{\Sigma} \sin^2 \theta d\varphi , \quad (3.27)$$

where $\Sigma = \sqrt{r^2 + a^2 + 2ra \cos \theta}$. With these expressions and the ones given in (3.16) – (3.18) it is straightforward to calculate $\vec{\omega}$ from (2.27). This can actually be done for any two-center solution. If we adjust the constants in the harmonic function in (3.24) so that $\vec{\omega} = 0$ for $\mathcal{G}_{1,2} = 0$, we obtain for a black hole in a Gödel-Taub-NUT spacetime

$$\vec{\omega} d\vec{x} = \left[\frac{1}{2} (n\mathcal{G}_1 - m\mathcal{G}_2) + (p^0\mathcal{G}_1 - q_0\mathcal{G}_2) \frac{R}{r} \right] r^2 \sin^2 \theta d\varphi . \quad (3.28)$$

The remaining part of the solution, namely ω_5 and f , is obtained by inserting the harmonic functions (3.24) into (2.28) and (2.32).

Next, we construct a black ring solution in a Gödel-Taub-NUT spacetime. The black ring was described by the harmonic functions in (3.15). To the harmonic function M we now

add the Gödel deformation $\mathcal{G}r \cos \theta$, so that

$$\begin{aligned} K^A &= \frac{p^A}{\Sigma} \quad , \quad L_A = h_A + \frac{q_A}{\Sigma} \quad , \\ M &= \mathcal{G}r \cos \theta - h_A p^A \left(1 - \frac{a}{\Sigma}\right) \quad , \quad N = n + \frac{1}{r} \quad , \\ \Sigma &= |\vec{x} - \vec{x}_0| = \sqrt{r^2 + a^2 + 2ra \cos \theta} \quad . \end{aligned} \quad (3.29)$$

In calculating $\vec{\omega}$ we use the relations given (3.25) – (3.27) as well as in (3.16) – (3.18), and we obtain (here we set $R = 1$)

$$\begin{aligned} \vec{\omega} d\vec{x} &= \mathcal{G} \left(\frac{n}{2} + \frac{p^0}{r} \right) r^2 \sin^2 \theta d\varphi \\ &+ h_A p^A \left([\cos \theta + 1] \left[1 - \frac{a+r}{\Sigma} \right] + na \left[\frac{r \cos \theta + a}{\Sigma} - 1 \right] \right) d\varphi \quad . \end{aligned} \quad (3.30)$$

Observe that in both cases the Gödel deformation does not affect the near horizon geometry, i.e. near the black hole at $r = 0$ and near the black ring at $r = a$, $\cos \theta = -1$ the Gödel deformation either vanishes or is constant. Therefore, the entropy of the black hole remains unaffected. On the other hand, since M grows linearly with r , also ω_5 grows with r and we have to face the problem of CTCs, as it happened in the overrotating case for black holes. In addition, also the four-dimensional solution can have CTCs, since the condition (2.47) will be violated with growing radial distance. On the other hand, if one has two Gödel deformations in M and N there is always a parameter choice so that the Gödel deformations in $\vec{\omega}$ cancel and CTC in four dimensions are avoided, which is obvious in the expression (3.28). More serious is the fact that the four-dimensional solution exhibits a curvature singularity at some finite radial distance which corresponds to the point where the circle along the ψ direction degenerates, i.e. where the condition (2.46) is violated and the solution becomes four-dimensional. As for the overrotating case, it would be interesting to discuss the effect of instanton corrections or higher derivative corrections. Observe that, with a growing harmonic function M , some of the scalar fields become small and therefore, the simplest correction to the prepotential in four dimensions which becomes important in this limit, is the term $\sim i\chi\zeta(3)(Y^0)^2$, see [75], where this term has been used as a regulator. Since χ is the Euler number of the internal space, this term encodes some of the higher derivative corrections in string theory. But before discussing effects of higher derivative corrections in more detail, let us mention that there is another (simple) possibility to avoid pathologies due to a growing function M . Namely, replacing the Gödel deformation in (3.22) with

$$\mathcal{G} z \rightarrow \mathcal{G} (1 - |z - z_0|) \quad , \quad (3.31)$$

yields an upper bound when introducing a source at $z = z_0$, which corresponds to a domain wall and is in the spirit of the discussion in [73]. A generalization of this would be a periodic

array of sources yielding an upper and lower bound for the function M . In doing so, one has however to keep in mind that these additional sources also contribute to the integrability constraint (2.7).

4 Three-charge BPS solutions and R^2 -corrections

Higher-order curvature corrections can convert an apparently pathological solution of General Relativity into a regular solution with an event horizon. This so-called cloaking of a singularity has recently been demonstrated to occur in string theory for certain two-charge black hole solutions in four dimensions [76, 77, 78, 79, 80, 48]. One example of such a two-charge solution is obtained in type IIA string theory on $K3 \times T_2$, by wrapping N_4 D4-branes on $K3$ and adding a gas of N_0 D0-branes to it. The resulting macroscopic entropy, which is entirely due to higher-curvature terms in the effective action, is found to be given by $\mathcal{S}_{\text{macro}} = 4\pi\sqrt{N_0 N_4}$ in the limit of large N_0, N_4 . This is in agreement with a counting of the microstates of the system [76].

The cloaking of singularities is not restricted to four dimensions. As shown in [48], R^2 -interactions in five (and higher) dimensions can also cloak the singularity of two-charge solutions in these dimensions. In the following, we will use the recently established connection between four and five-dimensional BPS solutions [33, 34, 44] to discuss the cloaking of five-dimensional two-charge solutions in a Taub-NUT geometry in terms of the cloaking of three-charge solutions in four dimensions. Here, the third charge is the Taub-NUT charge p^0 , which we take to be non-vanishing in order to be able to utilise the connection between four and five-dimensional BPS solutions.

Analysing the cloaking of five-dimensional singularities in terms of the four-dimensional solution has the advantage that in four dimensions one can rely on a precise algorithm for constructing the R^2 -corrected BPS solution. In five dimensions, on the other hand, there is not yet a clear understanding of the nature of the R^2 -interactions and their impact on five-dimensional BPS solutions.

R^2 -interactions lead to a departure from the Bekenstein-Hawking area law [81] for the macroscopic entropy of a black hole. In four dimensions, this departure is due to terms in the effective Wilsonian action associated with the supersymmetrisation of the square of the Weyl tensor [43]. On the other hand, the departure from the area law in four and five dimensions has been linked to a term in the effective action involving the Gauss-Bonnet combination [82, 47, 48]. Thus, it would appear that there are two combinations of R^2 -terms giving rise to the same leading correction to the entropy. Here we will show that

these two combinations are actually equal to one another when evaluated on the near-horizon solution. This may explain why the Gauss-Bonnet recipe manages to reproduce some of the corrections to the macroscopic entropy arising from a Wilsonian action with complicated R^2 -interactions.

Let us consider the near horizon geometry of a four-dimensional BPS black hole solution. This is a Bertotti-Robinson geometry, whose static line element we write as $ds^2 = -e^{2U} dt^2 + e^{-2U} d\vec{x}^2$ with $U = \log r + \text{const}$ and $r^2 = x^m x^m$. This is a maximally supersymmetric solution of the equations of motion of the Wilsonian $N = 2$ Lagrangian with R^2 -interactions. Let us evaluate the latter on this maximally supersymmetric solution. Most of the terms in the Lagrangian vanish when evaluated on this maximally supersymmetric background [30], and one is left with

$$8\pi e^{-1} \mathcal{L}|_{\text{BR}} = -\frac{1}{2} e^{-\kappa} R - \frac{i}{32} \left(F(X, \hat{A}) \bar{\hat{A}} - \text{h.c.} \right), \quad (4.1)$$

where $\hat{A} = (\varepsilon_{ij} T_{ab}^{ij})^2$ and $e^\kappa = G_4$ denotes Newton's constant in four dimensions.

On the solution, $F(X, \hat{A}) \bar{\hat{A}} = e^{4U} F(Y, \Upsilon) \bar{\Upsilon}$, where $\Upsilon = \bar{\Upsilon} = -64 U_m U_m$ and $U_m = \partial_m U$. Inserting this into (4.1) yields

$$8\pi e^{-1} \mathcal{L}|_{\text{BR}} = -\frac{1}{2} e^{-\kappa} R - 4 \text{Im} F(Y, \Upsilon) e^{4U} U_m U_m. \quad (4.2)$$

The holomorphic function $F(Y, \Upsilon)$ has an expansion of the form $F(Y, \Upsilon) = \sum_{g \geq 0} F^{(g)}(Y) \Upsilon^g$. Here, $F^{(0)}(Y)$ denotes the prepotential function of subsection 2.1. Let us now consider a particular function $F(Y, \Upsilon)$ of the form

$$F(Y, \Upsilon) = F^{(0)}(Y) + F^{(1)}(Y) \Upsilon, \quad (4.3)$$

and let us rewrite the term proportional to $F^{(1)}$ in (4.2) in terms of the Gauss-Bonnet combination evaluated on the solution. The Gauss-Bonnet combination GB can be written as $C^2 - 2R_{\mu\nu} R^{\mu\nu} + \frac{2}{3} R^2$, where C^2 denotes the square of the Weyl tensor. The latter vanishes for conformally flat solutions such as Bertotti-Robinson. Using $U_{mm} = U_m U_m = r^{-2}$ we note that $R = 2(-U_{mm} + U_m U_m) e^{2U}$ also vanishes (ignoring sources). Using $R_{tt} = -U_{mm} e^{4U}$ and $R_{mn} = -U_{pp} \delta_{mn} + 2U_m U_n$, we obtain $R_{\mu\nu} R^{\mu\nu} = 4(U_m U_m)^2 e^{4U}$. Therefore, we find that on the solution, (4.2) can be written as^c

$$e^{-1} \mathcal{L}|_{\text{BR}} = -\frac{1}{16\pi G_N} R - \frac{1}{2\pi} \text{Im} F^{(0)}(Y) e^{4U} U_m U_m - \frac{4}{\pi} \text{Im} F^{(1)}(Y) GB. \quad (4.4)$$

^cIn heterotic string theory, $F^{(1)} = -iS/64$ for large values of the dilaton S [83]. Inserting this into (4.4) and using $G_4 = 2$ yields precise agreement with the heterotic Lagrangian used in [48] to compute the entropy of small black holes.

Next we determine the correction to the Euclidean action due to the term proportional to $F^{(1)}$ in (4.4). The Euclidean solution is $H_2 \times S_2$ and has Euler character $\chi = (32\pi^2)^{-1} \int GB = 1 \times 2 = 2$. Using the fact that the scalar fields Y are constant in a Bertotti-Robinson space-time, we find that the $F^{(1)}$ -term in (4.4) contributes the following amount to the Euclidean action,

$$\Delta S_E = -256\pi \text{Im} F^{(1)}(Y) . \quad (4.5)$$

This we now compare with the corrections to the macroscopic entropy formula due to R^2 -interactions. The macroscopic entropy computed from the effective Wilsonian Lagrangian is given by [43]

$$\mathcal{S}_{\text{macro}} = \pi [|Z|^2 - 256 \text{Im} F_{\Upsilon}(Y, \Upsilon)] , \quad (4.6)$$

where here $\Upsilon = -64$ and $F_{\Upsilon} = \partial F / \partial \Upsilon$. For the function (4.3) this gives

$$\mathcal{S}_{\text{macro}} = \pi [|Z|^2 - 256 \text{Im} F^{(1)}(Y)] . \quad (4.7)$$

We therefore see that the correction to the Euclidean action (4.5) precisely equals the correction term proportional to $F^{(1)}$ in the macroscopic entropy (4.7). The latter is the Wald term which measures the deviation from the area law of Bekenstein and Hawking.

The above agreement suggests to view the Gauss-Bonnet recipe as an effective recipe which manages to capture some of the corrections to the entropy due to the complicated supersymmetrised R^2 -terms.

Next, let us discuss the cloaking of three-charge solutions in four dimensions. For convenience, we will consider solutions of heterotic string theory on $K3 \times T_2$. The four-dimensional R^2 -corrected effective Wilsonian action is known to contain a term $(S + \bar{S})^2 C_{\mu\nu\rho\sigma}^2$ at tree-level, where $C_{\mu\nu\rho\sigma}$ denotes the Weyl tensor and S the dilaton field. The tree-level holomorphic function $F(Y, \Upsilon)$ associated with a heterotic $N = 2$ compactification on $K3 \times T_2$ is given by

$$F(Y, \Upsilon) = -\frac{Y^1 Y^a \eta_{ab} Y^b}{Y^0} + c_1 \frac{Y^1}{Y^0} \Upsilon , \quad (4.8)$$

where we have suppressed instanton contributions. Here

$$Y^a \eta_{ab} Y^b = Y^2 Y^3 - \sum_{a=4}^n (Y^a)^2 , \quad a = 2, \dots, n , \quad (4.9)$$

with real constants $\eta_{ab} = \frac{1}{2} C_{ab}$ and $c_1 = -\frac{1}{64}$. The C_{ab} denote the intersection numbers of $K3$. The dilaton field is defined by $S = -iY^1/Y^0$. The moduli T^a are given by $T^a = -iY^a/Y^0$.

The Wilsonian $N = 2$ Lagrangian based on the holomorphic function $F(Y, \Upsilon)$ has supersymmetric charged multi-center solutions [30, 84]. The one-center solutions are static and spherically symmetric. The associated line element is given by $ds^2 = -e^{2U} dt^2 + e^{-2U} d\vec{x}^2$, where [30]

$$e^{-2U} = i \left[\bar{Y}^I F_I(Y, \Upsilon) - \bar{F}_I(\bar{Y}, \bar{\Upsilon}) Y^I \right] + 128i e^U \nabla^p \left[e^{-U} \nabla_p U (F_\Upsilon - \bar{F}_\Upsilon) \right] . \quad (4.10)$$

As discussed in subsection 2.1, the scalar fields Y^I ($I = 0, 1, \dots, n$) are determined in terms of an array of $2(n+1)$ harmonic functions (H^I, H_I) , given in (2.2), through the so-called generalised stabilisation equations [36, 30],

$$\begin{pmatrix} Y^I - \bar{Y}^I \\ F_I(Y, \Upsilon) - \bar{F}_I(\bar{Y}, \bar{\Upsilon}) \end{pmatrix} = i \begin{pmatrix} H^I \\ H_I \end{pmatrix} .$$

For a static solution, $H^I \nabla_p H_I - H_I \nabla_p H^I = 0$ and $\Upsilon = \bar{\Upsilon} = -64(\nabla_p U)^2$.

For a holomorphic function of the form (4.8) we have [83]

$$\begin{aligned} i \left[\bar{Y}^I F_I(Y, \Upsilon) - \bar{F}_I(\bar{Y}, \bar{\Upsilon}) Y^I \right] &= (S + \bar{S}) \left(\frac{1}{2} H_m^2 - 128c_1 (U')^2 \right) , \\ 128i e^U \nabla^p \left[e^{-U} \nabla_p U (F_\Upsilon - \bar{F}_\Upsilon) \right] &= 128c_1 \left[(S + \bar{S}) \left((U')^2 - U'' - \frac{2}{r} U' \right) - (S + \bar{S})' U' \right] , \end{aligned} \quad (4.11)$$

where $U' = dU/dr$ and $(S + \bar{S})' = d(S + \bar{S})/dr$. By combining these expressions we obtain

$$e^{-2U} = \frac{1}{2} (S + \bar{S}) H_m^2 - 128c_1 \left[(S + \bar{S}) \left(U'' + \frac{2}{r} U' \right) + (S + \bar{S})' U' \right] . \quad (4.12)$$

The real part of the dilaton field S , on the other hand, is determined by [83]

$$S + \bar{S} = 2 \sqrt{\frac{H_e^2 H_m^2 - (H_e \cdot H_m)^2}{H_m^2 [H_m^2 - 512c_1 (U')^2]}} , \quad (4.13)$$

where we have introduced the target-space duality invariant combinations

$$\begin{aligned} H_e^2 &= 2 \left(-H_0 H^1 + \frac{1}{4} H_a \eta^{ab} H_b \right) , \\ H_m^2 &= 2 \left(H^0 H_1 + H^a \eta_{ab} H^b \right) , \\ H_e \cdot H_m &= H_0 H^0 - H_1 H^1 + H_2 H^2 + \dots H_n H^n . \end{aligned} \quad (4.14)$$

Note that in the duality basis of perturbative heterotic string theory the electric H -vector is given by $(H_0, -H^1, H_2, \dots, H_n)$, whereas the magnetic H -vector reads $(H^0, H_1, H^2, \dots, H^n)$.

By combining (4.13) and (4.12) we obtain a non-linear differential equation for U . Similar non-linear differential equations have been recently discussed and solved in [78, 79, 80].

Linearising in c_1 still yields a complicated differential equation, namely

$$\begin{aligned} e^{-2U} &= \frac{1}{2}(S_0 + \bar{S}_0) H_m^2 - 128c_1 \left[(S_0 + \bar{S}_0) \left(U'' + \frac{2}{r}U' - 2(U')^2 \right) + (S_0 + \bar{S}_0)' U' \right] \\ &\quad + \mathcal{O}(c_1^2) , \end{aligned} \quad (4.15)$$

where

$$S_0 + \bar{S}_0 = 2 \sqrt{\frac{H_e^2 H_m^2 - (H_e \cdot H_m)^2}{H_m^2 H_m^2}} . \quad (4.16)$$

To utilise the 4D/5D-connection we take the harmonic functions H^I and H_I to be given as in (3.2). If the charges carried by the solution are generically non-vanishing, then the solution describes a one-center black hole solution with entropy given by [83]

$$\mathcal{S}_{\text{macro}} = -\frac{1}{2}\pi(S + \bar{S}) (p^2 + 512c_1) , \quad (4.17)$$

where S denotes the value of the dilaton field at the horizon. This value is determined by the attractor equations for S , which read [83]

$$\begin{aligned} q^2 - |S|^2 p^2 &= -2(S + \bar{S})(c_1 \Upsilon S + \text{h.c.}) , \\ 2ip \cdot q + (S - \bar{S})p^2 &= -2(S + \bar{S})(c_1 \Upsilon - \text{h.c.}) , \end{aligned} \quad (4.18)$$

where Υ takes the value -64 on the horizon. The combinations q^2, p^2 and $q \cdot p$ denote the following target-space duality invariant combinations of the charges [83],

$$\begin{aligned} q^2 &= 2q_0 p^1 - \frac{1}{2}q_a \eta^{ab} q_b , \\ p^2 &= -2p^0 q_1 - 2p^a \eta_{ab} p^b , \\ q \cdot p &= q_0 p^0 - q_1 p^1 + q_2 p^2 + \dots + q_n p^n . \end{aligned} \quad (4.19)$$

In the duality basis of perturbative heterotic string theory the electric charge vector is given by $(q_0, -p^1, q_2, \dots, q_n)$, whereas the magnetic charge vector reads $(p^0, q_1, p^2, \dots, p^n)$.

Inserting (4.18) into (4.17) yields the entropy

$$\mathcal{S}_{\text{macro}} = \pi \sqrt{q^2 p^2 - (q \cdot p)^2} \sqrt{1 + \frac{512c_1}{p^2}} . \quad (4.20)$$

This describes the R^2 -corrected entropy of the black hole with generic charges. Now consider restricting the charges to $(q_0 = 2J, q_A, p^0 = 1, p^A = 0)$ (where $A = 1, \dots, n$). Note that

$p^0 \neq 0$ is a necessary condition for the 4D/5D-connection [33, 34]. Then the entropy (4.20) becomes

$$\mathcal{S}_{\text{macro}} = 2\pi \sqrt{\frac{1}{4}q_1 q_a \eta^{ab} q_b - J^2} \sqrt{1 - \frac{256c_1}{q_1}}. \quad (4.21)$$

When $c_1 = 0$, this describes the entropy of a charged five-dimensional rotating BPS black hole in a Taub-NUT geometry. It is then tempting to conjecture that (4.20) describes the R^2 -corrected entropy of the five-dimensional BPS black hole in a Taub-NUT geometry. This is supported by the recent work [44].

In the absence of R^2 -interactions ($c_1 = 0$) the entropy (4.17) becomes equal to [85, 86]

$$\mathcal{S}_{\text{macro}} = \sqrt{q^2 p^2 - (q \cdot p)^2}. \quad (4.22)$$

This follows by inserting (4.18) into (4.17). Inspection of (4.22) shows that solutions with charges satisfying $p^2 = q \cdot p = 0$ have zero entropy in the absence of R^2 -interactions. However, in the presence of R^2 -interactions the entropy ceases to be vanishing, as can be seen from (4.17). These solutions therefore provide examples of black hole solutions which grow a horizon due to R^2 -interactions, thereby cloaking the singularity which is present in the absence of higher curvature interactions.

In the following we will be interested in solutions with $p^0 \neq 0$ so as to be able to utilise the 4D/5D connection. Then, demanding $p^2 = q \cdot p = 0, q^2 \neq 0$ results in $q_0 = q_1 = p^a = 0$. The solutions are therefore allowed to carry non-vanishing electric charges $(-p^1, q_2, \dots, q_n)$. The interpolating solution (4.12) is therefore constructed out of the following non-trivial harmonic functions

$$H^0 = n + \frac{p^0 R}{r}, \quad H^1 = h^1 + \frac{p^1 (RG_4)^{1/3}}{r}, \quad H_a = h_a + \frac{q_a (RG_4)^{2/3}}{Rr}, \quad (4.23)$$

whereas the remaining harmonic functions are constant, namely $H_0 = m, H_1 = h_1, H^a = h^a$. Note that the constraint $H^I \nabla_p H_I - H_I \nabla_p H^I = 0$ results in (here we set $G_4 = R^2$)

$$h^a q_a = m p^0 + h_1 p^1. \quad (4.24)$$

In the absence of R^2 -interactions ($c_1 = 0$), the four-dimensional solution has a naked singularity at $r = 0$. This can be seen from (4.15), which then reads

$$e^{-2U} = \sqrt{H_e^2 H_m^2 - (H_e \cdot H_m)^2}, \quad (4.25)$$

and which behaves as $r^{-3/2}$ at $r = 0$.

In the presence of R^2 -interactions, however, the solution grows a horizon. Inspection of (4.18) shows that the dilaton then takes the following value at the horizon,

$$S + \bar{S} = \sqrt{\frac{q^2}{-2c_1\Upsilon}} = \sqrt{-\frac{q_a\eta^{ab}q_b}{256c_1}}. \quad (4.26)$$

For this to be a positive quantity, the signs of the charges q_a have to be chosen in the appropriate way. The associated R^2 -corrected entropy reads [87]

$$\mathcal{S}_{\text{macro}} = -256c_1\pi(S + \bar{S}) = \pi\sqrt{-256c_1 q_a\eta^{ab}q_b}. \quad (4.27)$$

Thus we see that a three-charge black hole with charges (p^0, q_2, q_3) in four dimensions (or more generally a black hole with charges $(p^0, p^1, q_2, \dots, q_n)$) has a non-vanishing entropy which goes as $\sqrt{c_1}$, once R^2 -interactions are taken into account.

Evidence has been presented in [44] that the connection [33] between five-dimensional BPS solutions in a Taub-NUT space and four-dimensional BPS solutions continues to hold in the presence of R^2 -interactions. Using this connection, we conclude that the cloaking of the four-dimensional singularity of the three-charge solution also takes place in the five-dimensional solution when taking into account R^2 -effects. The cloaking of the five-dimensional singularity should be such that the entropy of the resulting five-dimensional two-charge black hole in the Taub-NUT geometry is given by (4.27).

The cloaking of five-dimensional singularities should not only apply to horizons with S^3 topology, but also to horizons with topology $S^1 \times S^2$, i.e. to black rings.^d After all, using the 4D/5D connection, black ring solutions descend to multiple center solutions in four dimensions [34]. The latter may, in the absence of R^2 -interactions, have multiple naked singularities which get cloaked by R^2 -interactions. This then implies a cloaking of black ring singularities in five dimensions. For instance, a two-center solution in four dimensions is connected to a five-dimensional black ring solution, if one of the centers (say at $r = 0$) carries the entire NUT charge p^0 , whereas the second center carries all the other charges [34]. Without R^2 -interactions, the second center is a naked singularity if the charges are restricted to satisfy $p^2 = p \cdot q = 0$. Since $p^0 = 0$ at this center, this implies that $q_1 p^1 = 0$. The second center is therefore allowed to carry electric charges $(q_0, -p^1, q_2, \dots, q_n)$. In the presence of R^2 -interactions the second center gets cloaked and its entropy is given by [87]

$$\mathcal{S}_{\text{macro}} = \pi\sqrt{512c_1 q^2} = \pi\sqrt{512c_1 (2q_0 p^1 - \frac{1}{2}q_a\eta^{ab}q_b)}. \quad (4.28)$$

^dThis has also been pointed out and studied in the recent paper [50]. Instanton corrections may, in principle, also contribute to the cloaking [65].

This should describe the entropy of a cloaked black ring in five dimensions. One may also consider other examples, for instance a four-dimensional two-center solution where one of the centers carries charges (p^0, q_2, q_3) , whereas the other center carries charges $(q_0, -p^1, q_2, q_3)$. In the absence of R^2 -interactions these two centers describe naked singularities. Turning on R^2 -interactions should then lead to a cloaking of the two naked singularities. In five dimensions, this would correspond to the cloaking of a black hole sitting at the center of a Taub-NUT geometry and of a black ring away from the center.

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